On Explicit Solutions of the Equations for Steady Compressible Flow

BRUCE L. HICKS
Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland
July 21, 1947

IN classical aerodynamics the steady flow of incompressible, inviscid fluids can often be described explicitly in terms of functions of a complex variable. There are, however, few accurate solutions of the equations for compressible flow that can be obtained in closed form. In view of the importance of obtaining more exact solutions, especially in the transonic region, the following outline of a new method of approach is submitted (see related work of Munk[1]). The flows considered are *adiabatic* but not necessarily isentropic and iso-energetic.

For uniplanar irrotational flow, the compressible flow equations can be written

$$\partial N/\partial\psi=\lambda_1(N)\partial\vartheta/\partial\varphi, \tag{1}$$

$$\partial N/\partial\varphi=-[\lambda_2(N)]^{-1}\partial\vartheta/\partial\psi, \tag{2}$$

where the velocity vector $\mathbf{V}=[g(N)RT]^{\frac{1}{2}}\mathbf{N}$; $|\mathbf{N}|=N=|\mathbf{V}|/[g(N)RT]^{\frac{1}{2}}$; ϑ is the angle of inclination of the vector $\mathbf{V}$ to the x axis; and φ and ψ are potential and stream functions for the vector $\mathbf{N}$, defined by the equations $\mathbf{N}=\nabla\varphi=\nabla\times\mathbf{k}\psi$. To a given choice of the function $g(N)$, there correspond flows with a given physical characteristic, namely, a relation between N and the pressure p on curves normal to the streamlines. Choice of $g(N)$ also fixes the form of the functions λ_1, λ_2 and all other functions of N or $\mathfrak{N}$ which will be introduced subsequently.

Except when $g(N)\propto(1-N^2)^{-1}$, irrotationality of $\mathbf{N}$ requires,[2] if the equation of motion is to be integrable, that (i) the stagnation pressure p_t vary according to the equations

$$\log p_t=\Psi(\psi)=\mathfrak{N}(N)-\Phi(\varphi) \tag{3}$$

(where Φ and Ψ are arbitrary), and is constant (between shocks) on streamlines; (ii) $\nabla^2\varphi\propto\mathfrak{L}(N)\Phi'(\varphi)$. If $g(N)\propto(1-N^2)^{-1}$, the integrability condition yields, instead of (3), the simple result $\nabla p_t\equiv 0$. The second of Eqs. (3) can, however, be assumed here also, in order that explicit integration of the flow equations can be achieved for this important case. It is possible, with other choices of $g(N)$, that the detailed analysis may be considerably simplified as it is known to be for diabatic flow.[2]

We now assume that (1) and (2) are integrable; then if $N(\varphi,\psi)$ can be found by some means, ϑ is calculable from

$$\vartheta=-\int_{\varphi,\psi_0}^{\varphi,\psi}\lambda_2(N)\frac{\partial N}{\partial\varphi}d\psi+\int_{\varphi_0,\psi_0}^{\varphi,\psi_0}[\lambda_1(N)]^{-1}\frac{\partial N}{\partial\psi}d\varphi. \tag{4}$$

Equation (3) together with assumed integrability of (1) and (2), demands that the functional-differential equation, involving the functions $\Phi(\varphi)$ and $\Psi(\psi)$, must be satisfied.

$$\mu_2(\mathfrak{N})F'(\Phi)+2\mu_2'(\mathfrak{N})F(\Phi)+\mu_1(\mathfrak{N})P'(\Psi)+2\mu_1'(\mathfrak{N})P(\Psi)=0, \tag{5}$$

where

$$F(\Phi)=(d\Phi/d\varphi)^{-\frac{1}{2}};\quad P(\Psi)=(d\Psi/d\psi)^{-\frac{1}{2}}.$$

After repeated integration with respect to Φ and Ψ, using (3), Eq. (5) can be reduced to a functional equation in $\Phi(\varphi)$ and $\Psi(\psi)$, which we shall schematically represent by

$$\mathfrak{F}[\Phi,\Psi,c_i(\varphi),k_j(\varphi)]=0, \tag{6}$$

in which $c_i(\varphi)$ and $k_j(\varphi)$ are functions introduced in the partial integrations.

If *any* functions $\Phi(\varphi)$ and $\Psi(\psi)$ can be found that reduce (6) to an identity, then (3) and (4) immediately yield explicit, closed expressions for N and ϑ. Further integration would yield x and y as functions of φ and ψ. Simple radial and vortex flows correspond to choosing Ψ and Φ, in turn, to be constant. No other exact solutions of the functional Eqs. (5) and (6) are known at present, although all previously known compressible flows have not been tested for their possible conformity to these equations.

The utility of the method, as compared to others, will, of course, depend on whether exact or approximate solutions of (5) and (6) can be found that yield flow patterns of some physical interest. There are indications that such solutions can be found in the transonic region. It is noted that the description $\mathbf{N}=\nabla\varphi$ applies to fields other than those of constant entropy and stagnation pressure, and may, therefore, make possible investigations of compressible flows down-stream from regions of heating and viscous dissipation.

This investigation was begun at the Cleveland laboratory of the N.A.C.A. in 1945. The assistance of the author's co-workers there, Dr. P. E. Guenther and Mr. R. H. Wasserman, is gratefully acknowledged.

[1] M. M. Munk, Phys. Rev. **72**, 176 (A) (1947).
[2] B. L. Hicks, Phys. Rev. **71**, 476 (A) (1947); B.R.L. Report No. 633, 1947. Also Hicks, Guenther, and Wasserman, Quart. App. Math. (October, 1947).

The Radial Density Variation of Gases and Vapors in a Centrifugal Field*

J. W. BEAMS
University of Virginia, Charlottesville, Virginia
July 21, 1947

IF the equation of state of a gas or vapor is expressed by the well-known relation

$$p=RT\left(\frac{d}{M}+B\frac{d^2}{M^2}+C\frac{d^3}{M^3}+\cdots\right)$$

where p is the pressure, T the absolute temperature, d the density, M the molecular weight, R the gas constant and where B, C, etc., are functions of T and are characteristic of the gas or vapor, then it can be shown theoretically that, under equilibrium conditions in a hollow centrifuge rotor with inside radius r spinning N r.p.s.,

$$\frac{2\pi^2MN^2r^2}{RT}=\log_e\frac{d_r}{d_0}+\frac{2B}{M}(d_r-d_0)+\frac{3C}{2M^2}(d_r^2-d_0^2)+\cdots, \tag{1}$$

where d_r and d_0 are the densities at the radius r and the axis, respectively. Since this latter equation may be used for determining the molecular weight of a gas or vapor, a precise experimental test of the relation has been undertaken. The apparatus consists of an air driven vacuum-type centrifuge which is similar to those previously described.[1] The rotor is a short accurately machined hollow cylinder

(2 cm high and 4.4 cm radius) with flat ends and is supported in the vacuum chamber by a thin flexible hollow shaft. The hollow shaft allows the pressure at the axis of the hollow spinning rotor to be measured at various rotor speeds. Measurements of this axis pressure were made at intervals of 100 r.p.s. up to 1200 r.p.s. with nitrogen, oxygen, and CO_2 gases, respectively. With these data, together with an accurate knowledge of the dimensions of the hollow rotor and auxiliary pressure measuring apparatus, the temperature, and the molecular weights of the respective gases, Eq. (1), could be tested. The results obtained show that this equation holds within the limit of experimental error. When Eq. (1) was used to determine the molecular weights of the nitrogen, oxygen, and CO_2 gases respectively, the values found agreed with the accepted values within considerably less than one percent.

Experiments also were carried out on pure ether vapor in equilibrium with the liquid phase in the periphery of the rotor. Ether was distilled into the evacuated rotor while it was spinning until about a cubic centimeter of the liquid condensed at the periphery. The pressure of the vapor at the axis was then measured as a function of the rotor speed and Eq. (1) was found to hold within the experimental error of roughly one percent. Consequently the molecular weight of a gas or vapor can be measured by the centrifuge method with a precision of about one percent. In the case of a radioactive gas or vapor, the molecular weight may be determined by measuring the amount of radioactivity at the axis of the rotor as a function of the rotor speed. The advantage of this centrifuge method for determining the molecular weights of radioactive gases or vapors is that only an extremely minute amount of the radioactive material is required, especially when the material can be mixed with a gas such as CO_2, nitrogen or helium.

* This work was supported by the Bureau of Ordnance, U. S. Navy under Contract NOrd-7873.

[1] J. W. Beams, Rev. Mod. Phys. 10, 249 (1938).

The Motion of a Particle in an Electromagnetic and Gravitational Field

H. C. Corben and Mulaika Corben
Carnegie Institute of Technology, Pittsburgh, Pennsylvania
July 21, 1947

FITTING gravitational theory into the framework of special relativity theory is a problem that has proved practically insoluble. At the same time, the incorporation of electromagnetic phenomena into general relativity theory has proved extremely complex. We want to point out that the equations of motion for a particle in an electromagnetic and gravitational field may be derived from a simple Lagrangian which is based on an extended principle of special relativity theory.

We consider the position and velocity space of the particle $q^\alpha=(x^\mu,\ ilv^\mu)$, $(\alpha=1\cdots 8,\ \mu=1\cdots 4)$ where l is a constant of the dimensions of length, and $v^\mu=\dot{x}^\mu$, a dot denoting differentiation with respect to $ds=(dx^\mu dx_\mu)^{\frac{1}{2}}$. For any four-vector $B^4=B_4$, $B^j=-B_j(j=1,2,3)$. We now define an 8-invariant Lagrangian

$$L=A_\alpha dq^\alpha/d\tau \quad (\alpha=1\cdots 8)$$

where $d\tau=(dq_\alpha dq^\alpha)^{\frac{1}{2}}$ and A_α is an 8-vector describing the interaction between the particle and the field. We write $A_\alpha=(e\kappa_\mu,\ iml^{-1}\epsilon_\mu)$ where κ_μ, ϵ_μ are two four-vectors and e, m are constants characteristic of the particle. Then

$$\mathfrak{L}=Ld\tau/ds=e\kappa_\mu v^\mu-m\epsilon_\mu\dot{v}^\mu.$$

Hence we define[1]

$$p_\mu=\frac{\partial\mathfrak{L}}{\partial v^\mu}-\frac{d}{ds}\left(\frac{\partial\mathfrak{L}}{\partial\dot{v}^\mu}\right),$$
$$ilp_{\mu+4}=-P_\mu=\partial\mathfrak{L}/\partial\dot{v}^\mu.$$

The variational principle

$$\delta\int Ld\tau=\delta\int\mathfrak{L}ds=0$$

then yields $\dot{p}_\mu=\partial_\mu\mathfrak{L}$, where $\partial_\mu\equiv\partial/\partial x^\mu$.

In general the A_α may depend upon both the position and velocity coordinates, but we consider here only the special case in which they are functions of the x^μ only. Then

$$p_\mu=m\dot{\epsilon}_\mu+e\kappa_\mu;\quad P_\mu=m\epsilon_\mu;$$
$$\dot{p}_\mu=ev^\nu\partial_\mu\kappa_\nu-m\dot{v}^\nu\partial_\mu\epsilon_\nu.$$

The Hamiltonian may be defined by

$$H=p_\alpha(dq^\alpha/d\tau)-L=(p_\alpha-A_\alpha)(dq^\alpha/d\tau)$$

so that

$$dq^\alpha/d\tau=\partial H/\partial p_\alpha;\quad dp_\alpha/d\tau=-\partial H/\partial q^\alpha.$$

Using $d/ds\equiv v^\nu\partial_\nu$ for the special case considered here, we may rewrite the equations of motion thus

$$d/ds[v^\nu(\partial_\mu\epsilon_\nu+\partial_\nu\epsilon_\mu)]=(e/m)v^\nu(\partial_\mu\kappa_\nu-\partial_\nu\kappa_\mu)+v^\nu v^\sigma\partial_\mu\partial_\sigma\epsilon_\nu.$$

We now define

$$g_{\mu\nu}=\partial_\mu\epsilon_\nu+\partial_\nu\epsilon_\mu;\quad f_{\mu\nu}=\partial_\mu\kappa_\nu-\partial_\nu\kappa_\mu,$$

so that $g_{\mu\nu}$, $f_{\mu\nu}$ are 4-tensors, respectively symmetrical and antisymmetrical, derivable from the 8-vector A_α. Thus

$$g_{\mu\nu}(dv^\nu/ds)=(e/m)f_{\mu\nu}v^\nu+\tfrac{1}{2}v^\nu v^\sigma(\partial_\mu g_{\nu\sigma}-\partial_\sigma g_{\mu\nu}-\partial_\nu g_{\mu\sigma}),$$

which suggests that we may interpret the $f_{\mu\nu}$ as the electromagnetic field strengths and the $g_{\mu\nu}$ as the gravitational potentials. For the general case in which the A_α involves v^μ explicitly, other types of fields appear in these equations as well.

Alternatively we have

$$\mathfrak{L}+m(d/ds)(\epsilon_\mu v^\mu)=e\kappa_\mu v^\mu+\tfrac{1}{2}mg_{\mu\nu}v^\mu v^\nu$$

from which, clearly, the equations of motion also follow. We also have $\mathfrak{H}=H(d\tau/ds)\equiv\frac{1}{2}mg_{\mu\nu}v^\mu v^\nu$. We note that for $\epsilon_\mu=\frac{1}{2}x_\mu$, $g_{ii}=-1$, $g_{44}=1$, $g_{\mu\nu}=0(\mu\neq\nu)$; then if $\mathcal{P}_\mu=p_\mu+\dot{P}_\mu$, it follows that $\mathcal{P}_\mu=mv_\mu+e\kappa_\mu$, $d\mathcal{P}_\mu/dt=ev^\nu\partial_\mu\kappa_\nu$, the usual electromagnetic equations.

The above results suggest that the equations of motion of a particle are covariant for arbitrary rotations of its position and velocity space. Such rotations are found to relate observers who are spinning or accelerated with respect to each other, and clearly mix the electromagnetic and gravitational components of A_α. Analysis of this group of rotations and of the field equations for A_α will be published soon.

[1] E. T. Whittaker, *Analytical Dynamics* (Dover Press, New York, 1944), p. 266.